

Diagonal Transformations and Cycle Parities of Quadrangulations on Surfaces

ATSUHIRO NAKAMOTO

*Department of Mathematics, Faculty of Education, Yokohama National University,
156 Tokiwadai, Hodogaya-ku, Yokohama 240, Japan*

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In this paper, we shall show that any two quadrangulations on any closed surface can be transformed into each other by diagonal slides and diagonal rotations if they have the same and sufficiently large number of vertices and if the homological

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1. INTRODUCTION

A quadrangulation G on a closed surface F^2 is a simple graph embedded in F^2 so that each face of G is quadrilateral. The diagonal slide and the diagonal rotation were defined in [1] as two transformations of quadrangulations. See Fig. 1. We also call the both transformations *diagonal transformations* in total. If the graph obtained by a diagonal slide is not a simple graph, then we do not apply it. If two quadrangulations G_1 and G_2 on a closed surface F^2 can be transformed into each other by diagonal transformations, then G_1 and G_2 are said to be *equivalent* to each other. The quadrangulations on a closed surface, except for the sphere, fall into two classes, in which one is bipartite and the other is non-bipartite. Note that there does not exist a non-bipartite quadrangulation on the sphere. The author has shown the following theorem.

THEOREM 1 (A. Nakamoto [1]). *There exists a positive integer $N(F^2)$ for any closed surface F^2 such that any two bipartite quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq N(F^2)$ are equivalent to each other, up to homomorphism.*

This theorem describes about only bipartite case. Actually, there exists a pair of inequivalent non-bipartite quadrangulations on the Klein bottle with the same and arbitrarily large number of vertices [1]. However, in the projective plane [4] and the torus [3], it has been shown that any two non-bipartite quadrangulations with the same number of vertices are

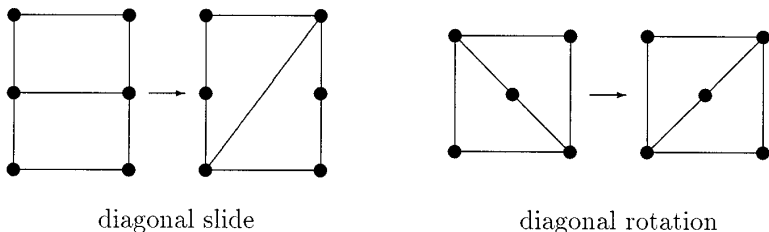


FIG. 1. The diagonal slide and the diagonal rotation.

equivalent, up to homeomorphism. When are two given quadrangulations equivalent to each other in general?

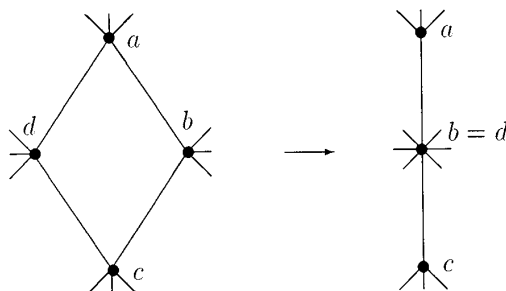
It is easy to see that any two cycles of a quadrangulation on a closed surface F^2 have the same length modulo 2 if they are homotopic to each other on F^2 . Two quadrangulations G_1 and G_2 on a closed surface F^2 are said to *have the same cycle parity* if for each closed curve l on F^2 , a cycle C_1 of G_1 and a cycle C_2 of G_2 both of which are homotopic to l on F^2 have the same length modulo 2. We can also see that both of a diagonal slide and a diagonal rotation change the set of cycles of a quadrangulation on a closed surface F^2 but preserve the parity of length of cycles with the same homotopy type on F^2 . Thus, if two quadrangulations are equivalent to each other, up to isotopy, then they must have the same cycle parity.

In this paper, we shall extend Theorem 1 to show Theorem 2. Since two bipartite quadrangulations have the same cycle parity, Theorem 2 implies Theorem 1.

THEOREM 2. *For any closed surface F^2 , there exists a positive integer $M(F^2)$ such that any two quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq M(F^2)$ are equivalent to each other, up to homeomorphism, if they have the same cycle parity.*

2. BIPARTITE CASE

Let G be a quadrangulation on a closed surface F^2 and f a face of G . And let a, b, c and d be vertices on the cycle bounding f in this order. A *face contraction* of f at $\{b, d\}$ is defined to identify b and d and replace two pairs of multiple edges ab and bc with two single edges respectively, as shown in Fig. 2. If b and d are joined or are adjacent with a common vertex v ($\neq a, c$), then a face contraction of f at $\{b, d\}$ yields a loop or multiple edges and destroys the simpleness of G . In this case, we do not apply the face contraction. If we can apply a face contraction of f , then f is said to be *contractible*. Note that there are two ways to contract a face since each

FIG. 2. A face contraction at $\{b, d\}$.

face has two diagonal pairs of vertices. If a quadrangulation T on F^2 is obtained from G by a sequence of face contractions, then G is said to be *contractible* to T . A quadrangulation on F^2 which is contractible to no other quadrangulation is said to be *irreducible*.

Theorem 1 has been proved by using the following four propositions. Among them, Propositions 4 and 5 play important roles to prove Theorem 2. So, we outline the proof of them. See [1] for detail.

PROPOSITION 3. *The number of irreducible bipartite quadrangulations on any closed surface F^2 is finite, up to homeomorphism.*

PROPOSITION 4. *A vertex of degree 2 of a quadrangulation G can be moved into any face of G by a sequence of diagonal slides.*

Proof. It is easy to see that a sequence of two diagonal slides moves a vertex of degree 2 into an adjacent face as shown in Fig. 3. By repeating this operation, we can move a vertex of degree 2 to any face. ■

Let G be a quadrangulation on a closed surface F^2 . Let Γ_n denote a quadrilateral region containing n vertices of degree 2 as shown in Fig. 4. And let $G + \Gamma_n$ denote a quadrangulation on F^2 obtained from G by adding Γ_n into a face of G . Here, $G + \Gamma_n$ represents various quadrangulations

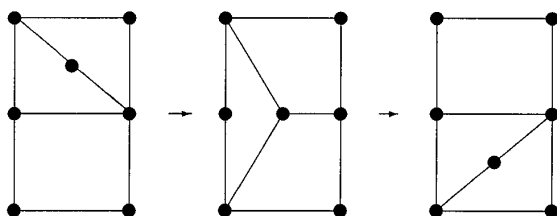
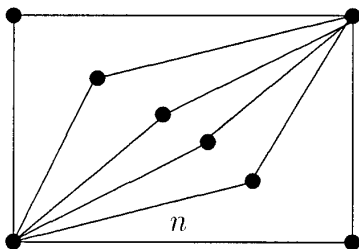


FIG. 3. Moving a vertex of degree 2.

FIG. 4. Γ_n .

depending on our choice of a face of G to add Γ_n . However, by Proposition 4, $G + \Gamma_n$ denotes a unique quadrangulation, up to equivalence.

PROPOSITION 5. *Let G and T be quadrangulations on a closed surface F^2 . If G is contractible to T , then G is equivalent to $T + \Gamma_n$, where $m = |V(G)| - |V(T)|$.*

Proof. Let $G = G_0, G_1, \dots, G_m = T$ be a sequence of quadrangulations such that G_{l+1} is obtained from G_l by a face contraction for $0 \leq l \leq m-1$. It is clear that $m = |V(G)| - |V(T)|$. Let u be a vertex of G_l and let $v_0 v_1 \dots v_{n-1}$ be the cycle surrounding the union of faces which meet u , where $n = 2 \deg(u)$ and v_1, v_3, \dots, v_{n-1} are adjacent to u . Suppose that the contraction of the face f bounded by $uv_1 v_0 v_{n-1}$ at $\{u, v_0\}$ yields G_{l+1} .

Since f is contractible at $\{u, v_0\}$, each v_i ($i = 3, 5, \dots, n-3$) neither is adjacent to nor coincides with v_0 . Otherwise, G_{l+1} would have multiple edges between v_0 and v_i or a loop incident with v_i , a contradiction. Thus, in G_l , a diagonal slide can replace an edge uv_i with $v_0 v_{i+2}$ for $i = 1, 3, \dots, n-5$ in this order (without losing the simpleness of a graph), and hence we get a quadrangulation G' such that v_0 is adjacent to v_1, v_3, \dots, v_{n-1} and that u has only two neighbors v_{n-1} and v_{n-3} . This is nothing but $G_{l+1} + \Gamma_1$. Thus, G_l is equivalent to $G_{l+1} + \Gamma_1$. Therefore, we can obtain inductively that G is equivalent to

$$T + \underbrace{\Gamma_1 + \Gamma_1 + \dots + \Gamma_1}_m.$$

Here, by Proposition 4, we can see that G is equivalent to $T + \Gamma_m$. Thus, the proposition follows. ■

Two quadrangulations G_1 and G_2 on a closed surface F^2 are said to be *stably equivalent* to each other if there exists a pair of non-negative integers m_1 and m_2 such that $G_1 + \Gamma_{m_1}$ and $G_2 + \Gamma_{m_2}$ are equivalent. Note that G_1 and G_2 are not assumed to have the same number of vertices.

PROPOSITION 6. *Any two bipartite quadrangulations on F^2 are stably equivalent to each other.*

Propositions 4 and 5 follow in general quadrangulations, but Propositions 3 and 6 could be actually proved, depending on a property of bipartite graphs. Thus, in order to show Theorem 2, we shall improve Propositions 3 and 6 so that they follow in general quadrangulations.

3. PROOF OF THEOREM 2

At first, we shall show the finiteness of the number of irreducible quadrangulations on any closed surface. In [2], it has been shown that the number of vertices of irreducible quadrangulations on a closed surface F^2 is bounded by a linear function of a genus of F^2 .

THEOREM 7 (A. Nakamoto and K. Ota [2]). *Let G be an irreducible quadrangulation on a closed surface F^2 and $r := 2 - \chi(F^2) > 0$, where $\chi(F^2)$ denotes the Euler characteristic. Then,*

$$|V(G)| \leq 186r - 64.$$

THEOREM 8. *There exist finitely many irreducible quadrangulations on any closed surface.*

Now we shall show the stable equivalence of any two quadrangulations with the same cycle parity. Let G be a quadrangulation on a closed surface F^2 . A quadrangulation G' on F^2 is said to be a *refinement* of G if G' contains a subdivision of G in which each edge of G is subdivided by an even number of vertices. Observe that G and G' have the same cycle parity since each cycle C of G and the cycle of G' corresponding to C have the same parity of length. Thus, if two quadrangulations G_1 and G_2 on a closed surface F^2 do not have the same cycle parity then there exists no common refinement of G_1 and G_2 .

LEMMA 9. *If G_1 and G_2 are two quadrangulations on a closed surface F^2 which have the same cycle parity, then there exists a common refinement G of G_1 and G_2 .*

Proof. Embed G_1 and G_2 in F^2 simultaneously so that G_1 and G_2 intersect only on their edges. Put a vertex on each intersection of $G_1 \cup G_2$. Let \tilde{G} be the resulting graph. Subdividing edges of \tilde{G} and adding edges to be quadrangular, we shall construct a common refinement G of G_1 and G_2 .

By the definition of a refinement, each edge of $G_i (i=1, 2)$ must be subdivided by an even number of vertices, that is, must be an odd-length

path in G . Also, since G is a quadrangulation, each face of \tilde{G} must be a 2-cell bounded by a cycle of even length in G . Thus, we shall show that by subdividing each edge of \tilde{G} by one or no vertex, we can construct a subdivision of \tilde{G} contained in G . To do so, we have only to solve a system of linear equations over \mathbf{Z}_2 , that is, we find $x_{e_i} \in \mathbf{Z}_2$ for each $e_i \in E(\tilde{G})$ ($i = 1, \dots, n$) such that

$$\sum_{e_i \subseteq s_j} x_{e_i} = 1 \quad \text{for each } s_j \in E(G_1) \quad (j = 1, \dots, \alpha), \quad \dots \quad (1)$$

$$\sum_{e_i \subseteq t_j} x_{e_i} = 1 \quad \text{for each } t_j \in E(G_2) \quad (j = 1, \dots, \beta), \quad \dots \quad (2)$$

$$\sum_{e_i \subset f_j} x_{e_i} = 0 \quad \text{for each face } f_j \text{ of } \tilde{G} \quad (j = 1, \dots, \gamma), \quad \dots \quad (3)$$

where α and β are the number of edges of G_1 and G_2 , respectively, and γ is the number of faces of \tilde{G} .

Expressing this system of linear equations by using a matrix, we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1l} & \cdots & a_{1n} \\ \vdots & b & \ddots & \vdots & \ddots & \vdots \\ a_{\alpha 1} & a_{\alpha 2} & \cdots & a_{\alpha l} & \cdots & a_{\alpha n} \\ \hline b_{11} & b_{12} & \cdots & b_{1l} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{\beta 1} & b_{\beta 2} & \cdots & b_{\beta l} & \cdots & b_{\beta n} \\ \hline c_{11} & c_{12} & \cdots & c_{1l} & \cdots & c_{1n} \\ \vdots & b & \ddots & \vdots & \ddots & \vdots \\ c_{\gamma 1} & c_{\gamma 2} & \cdots & c_{\gamma l} & \cdots & c_{\gamma n} \end{pmatrix} \begin{pmatrix} x_{e_1} \\ x_{e_2} \\ x_{e_3} \\ x_{e_4} \\ \vdots \\ x_{e_l} \\ \vdots \\ x_{e_n} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where each a_{ij} , b_{ij} and c_{ij} for any i and j is a coefficient in the equations (1), (2) and (3) respectively, and equal to 0 or 1 respectively. We denote row-vectors (a_{i1}, \dots, a_{in}) , (b_{j1}, \dots, b_{jn}) , (c_{k1}, \dots, c_{kn}) by \mathbf{a}_i , \mathbf{b}_j , \mathbf{c}_k , respectively.

Now suppose that such x_{e_i} 's do not exist. Then, by the argument of linear algebra, there exist index sets $I \subseteq \{1, \dots, \alpha\}$, $J \subseteq \{1, \dots, \beta\}$ and $K \subseteq \{1, \dots, \gamma\}$ such that

$$\sum_{i \in I} \mathbf{a}_i + \sum_{j \in J} \mathbf{b}_j + \sum_{k \in K} \mathbf{c}_k = \mathbf{0}$$

and such that

$$\sum_{i \in I} \mathbf{a}_i \mathbf{x} + \sum_{j \in J} \mathbf{b}_j \mathbf{x} + \sum_{k \in K} \mathbf{c}_k \mathbf{x} = \mathbf{1}$$

for $\mathbf{x} = {}^T(x_{e_1}, \dots, x_{e_n})$. Thus $C_1 := \{s_i \mid i \in I\}$, $C_2 := \{t_j \mid j \in J\}$ and $F' := \{f_k \mid k \in K\}$ satisfy the two conditions

- (i) for any $e_i \in E(\tilde{G})$, e_i is contained in $C_1 \cup C_2$ if and only if e_i is incident with an odd number of faces in F' , and
- (ii) $|C_1| + |C_2|$ is odd,

since each $e_i \in E(\tilde{G})$ must be contained exactly once in either G_1 or G_2 and since each edge is incident with at most two faces. The condition (i) implies that the boundary of $\bigcup_{k \in K} f_k$ coincides with $C_1 \cup C_2$ and hence C_1 and C_2 are homologous on F^2 . However, since G_1 and G_2 have the same cycle parity, C_1 and C_2 must have the same length modulo 2 and hence this situation contradicts to the condition (ii). Thus, the above system of linear equations has a solution. Therefore, the lemma follows. ■

LEMMA 10. *Let G be a quadrangulation on a closed surface F^2 and G' a refinement of G obtained by adding two vertices on an edge of G and adding two edges. Then G' is contractible to G .*

Proof. Let f and h be two faces of G sharing an edge $e = v_2v_5$ and let $v_1v_2v_3v_4v_5v_6$ be the cycle bounding $f \cup h$. Add two vertices x, y on e and add two edges to get G' . There are three possibilities, up to symmetry, as shown in Fig. 5. In each case, G can be obtained from G' by the face contraction at $\{y, v_4\}$ followed by one at $\{x, v_5\}$. ■

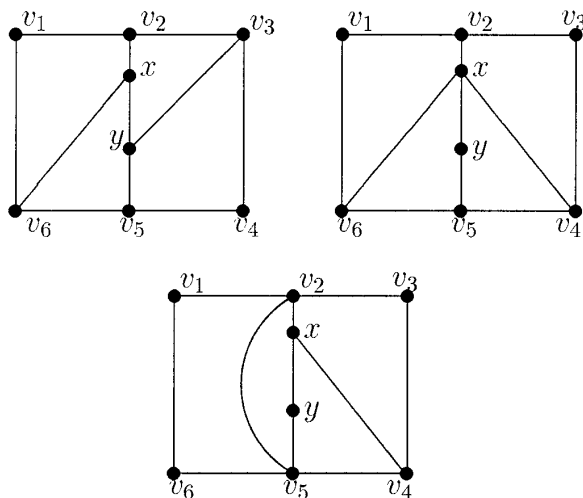


FIGURE 5

THEOREM 11. *Any two quadrangulations on a closed surface F^2 are stably equivalent to each other if they have the same cycle parity.*

Proof. Let G_1 and G_2 be quadrangulations on a closed surface F^2 which have the same cycle parity. There exists a common refinement G of G_1 and G_2 , by Lemma 9. Unfortunately, G might not be contractible to G_1 or G_2 . So, we shall show that there is another refinement \tilde{G} of G which is contractible to G and G_1 in common. Let G'_1 be the subdivision of G_1 contained in G . A 2-cell region of G (bounded by a cycle of even length) which was a face of G'_1 is said to be a *base region*. We may assume that each base region of G has no *chord*, that is, there is no edge uv of G such that $u, v \in V(G'_1)$ and $uv \notin E(G'_1)$. (Otherwise, subdivide a chord by an even number of vertices. And re-define G afterward.)

Let F be a base region of G corresponding to a face of G_1 . Let $L = b_1 w_1 b_2 w_2 \cdots b_n w_n$ be an odd-length path of the boundary cycle of F corresponding to an edge e of G_1 . We carry out the following deformation at each w_i . Let u_1, \dots, u_k be the neighbors of w_i in $\text{Int } F$ in this order and let $w_i b_i v_1 u_1, w_i u_1 v_2 u_2, \dots, w_i u_{k-1} v_k u_k, w_i u_k v_{k+1} b_{i+1}$ be the faces of G in F meeting at w_i . First subdivide an edge $w_i u_j$ to be a path $w_i y_j x_j u_j$ of length 3 ($j=1, \dots, k$). And next add new edges $b_i x_1, y_k v_{k+1}, y_{j-1} x_j, x_{j-1} u_j$ ($j=2, \dots, k$). See Fig. 6.

The resulting quadrangulation \tilde{G} is contractible to G , by Lemma 10. Moreover, we can apply face contractions at $\{b_i, y_1\}, \{y_1, y_2\}, \dots, \{y_{k-1} y_k\}, \{y_k, b_{i+1}\}$ or at $\{b_i, b_{i+1}\}$ for all w_i at a time, so that b_1 and w_n are joined by an edge afterward. Notice that no chord arises in these operations. Thus, F can be contractible to a quadrilateral region which is not a face. Here, by Lemma 3 in [1], a quadrilateral region which is not a face

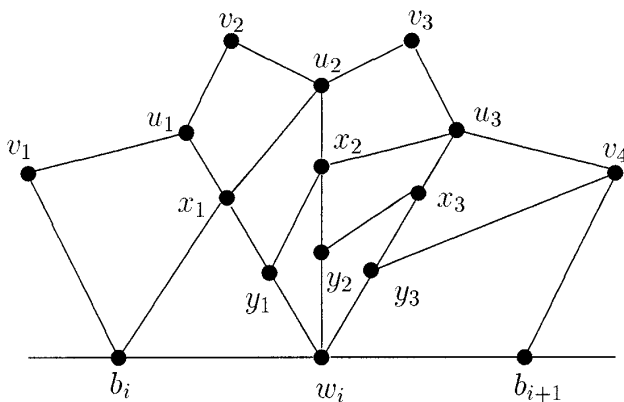


FIGURE 6

contains a contractible face and hence F can be contractible to a face. So, \tilde{G} is contractible to G_1 . Thus, \tilde{G} is contractible to G and G_1 in common. Similarly, we can construct a common refinement \tilde{G}' of G_2 and G by the operation in Lemma 10.

Now suppose that $|V(\tilde{G})| \geq |V(\tilde{G}')|$ without loss of generality. By Proposition 5, \tilde{G} , $G_1 + \Gamma_a$, and $G + \Gamma_b$ are equivalent and G' , $G_2 + \Gamma_c$ and $G + \Gamma_d$ are also equivalent, where a , b , c and d denote the numbers of new vertices of degree 2. Hence we can see that $G_1 + \Gamma_a$, $G + \Gamma_b$, $+ \Gamma_{d+l}$ and $G_2 + \Gamma_{c+l}$ are all equivalent one another, where $l = |V(\tilde{G})| - |V(\tilde{G}')|$. Therefore, G_1 and G_2 are stably equivalent to each other. ■

We have now prepared to prove Theorem 2.

Proof of Theorem 2. Let G be a quadrangulation on a closed surface F^2 . By Theorem 8, the number of irreducible quadrangulations on F^2 which have the same cycle parity as G is finite up to homeomorphism. Let T_1, \dots, T_k be all of such irreducible quadrangulations on F^2 . All of T_i ($1 \leq i \leq k$) are stably equivalent to one another by Theorem 11. In particular, there exists a positive integer M such that for any i and j ($1 \leq i, j \leq k$), $T_i + \Gamma_{n_i}$ and $T_j + \Gamma_{n_j}$ are equivalent to each other with $|T_i| + n_i = |T_j| + n_j = M$.

Suppose that $|V(G)| \geq M$. Since G is contractible to one of T_i ($1 \leq i \leq k$), say T_l , we know by Proposition 5, that G can be transformed into $T_l + \Gamma_{n'}$, where $n' = |V(G)| - |V(T_l)|$. Since $n_l \leq n'$ and since $T_1 + \Gamma_{n_1}$ and $T_l + \Gamma_{n_l}$ are equivalent, we have that $T_1 + \Gamma_{n'_1}$ and $T_l + \Gamma_{n'}$ are equivalent to G with $n'_1 = n_1 + n' - n_l$. Therefore, any two quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq M$ which have the same cycle parity are equivalent to each other via the standard form $T_1 + \Gamma_{n'_1}$. ■

In [4], it has been shown that, in non-bipartite quadrangulations, the diagonal rotation can be realized by a sequence of diagonal slides. So, in Theorem 2, the diagonal rotation is needed only in case that both G_1 and G_2 are bipartite.

Introducing a more loose equivalence of cycle parities, we can easily obtain the following corollary. Two quadrangulations G_1 and G_2 on a closed surface F^2 are said to have *congruent* cycle parities if there exists a homeomorphism $h: F^2 \rightarrow F^2$ such that G_1 and $h(G_2)$ have the same cycle parity.

COROLLARY 12. *For any closed surface F^2 , there exists a positive integer $M'(F^2)$ such that any two quadrangulations G_1 and G_2 on F^2 with $|V(G_1)| = |V(G_2)| \geq M'(F^2)$ are equivalent to each other, up to homeomorphism, if and only if they have congruent cycle parities.*

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